Exam. Code : 211002
Subject Code : 4977

M.Sc. (Mathematics) 2nd Semester

ALGEBRA—II

Paper—MATH-563

Time Allowed—2 Hours] [Maximum Marks—100

Note :— There are ***eight*** questions of equal marks. Candidates are required to attempt any ***four*** questions.

1. (a) Define prime ideal of ring R with unity. Find all prime ideals of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. Further find all prime ideals of $\mathbb{Z}_{100}$.

   (b) Let M be Abelian group and End (M) be the set of endomorphisms of M. Prove that End (M) is a ring.

2. (a) Let *l* be a 2-sided ideal of ring R. Write a detailed note on construction of quotient ring R/l. Is the same construction possible if 2-sided ideal is replaced by subring ? Justify.

(b) Prove that if $\phi : R \rightarrow R'$ is a ring homomorphism and l is an ideal of $R \subseteq \text{Ker } \phi$, then ϕ induces a ring homomorphism from $R/l \rightarrow R'$.

3. (a) Prove that every integral domain can be embedded in a field.

(b) Prove that the ideal $A = (a_0)$ is maximal in Euclidean ring if and only if a_0 is prime element of R.

4. (a) Prove that if R is UFD, then product of two primitive polynomials in R[x] is primitive.

(b) State and prove Eisenstein's Criteria for irreducibility of polynomials.

5. (a) Show that $x^3 - 2x - 2$ is irreducible over $\mathbb{Q}$. Let θ be a root. Compute $(1 + \theta)(1 + \theta + 0^2$ and $\frac{1+\theta}{1+\theta+\theta^2}$ in $Q(\theta)$.

(b) Define finitely generated field extension. Find necessary and sufficient condition for a finitely generated extension to be algebraic.

6. (a) Find the splitting field of K of $x^p - 1$ over $\mathbb{Q}$. Hence find $[K : \mathbb{Q}]$.

(b) Prove that finite separable extension of a field is simple.

7. (a) Prove that for a given prime p and positive integer n, there exist a field of order p^n. Any two such fields are isomorphic.

(b) Prove that if real number α is obtained from a field $F \subset \mathbb{R}$ by a series of ruler and compass construction then $[F(\alpha) : F]$ = power of 2.

8. State and prove Fundamental Theorem of Galois Theory.